Exercise 56

In the theory of relativity, the Lorentz contraction formula

$$L = L_0\sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v\to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

Solution

Take the limit of L as $v \to c^-$.

$$\lim_{v\to c^-} L = \lim_{v\to c^-} L_0\sqrt{1 - \frac{v^2}{c^2}}$$

The limit of a constant times a function is the constant times the limit of the function.

$$\lim_{v\to c^-} L = L_0 \lim_{v\to c^-} \sqrt{1 - \frac{v^2}{c^2}}$$

Use the Root Law.

$$\lim_{v\to c^-} L = L_0\sqrt{\lim_{v\to c^-}\left(1 - \frac{v^2}{c^2}\right)}$$

The limit of a difference is the difference of the limits.

$$\lim_{v\to c^-} L = L_0\sqrt{\lim_{v\to c^-} 1 - \lim_{v\to c^-} \frac{v^2}{c^2}}$$

The limit of a quotient is the quotient of the limits, provided that the limit of the denominator is not zero.

$$\lim_{v\to c^-} L = L_0\sqrt{\lim_{v\to c^-} 1 - \frac{\lim\limits_{v\to c^-} v^2}{\lim\limits_{v\to c^-} c^2}}$$

The limit of a product is the product of the limits.

$$\lim_{v\to c^-} L = L_0\sqrt{\lim_{v\to c^-} 1 - \frac{\left(\lim\limits_{v\to c^-} v\right)^2}{\lim\limits_{v\to c^-} c^2}}$$

$$= L_0\sqrt{1 - \frac{(c)^2}{c^2}}$$

$$= 0$$

A left-hand limit is necessary because the velocity cannot be faster than the speed of light.